Geometric Electro-Magnetic Field Equations for Particles with Charge and Spin

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The complete and self-consistent "Maxwell" equations are displayed algebraically, describing (time-delayed) force and torque interactions between moving particles with Charge and Spin.

The Point

The algebra is the **4-level Grassmann linear algebra** describing 3D Euclidean space, with 4 Basis Elements **{ 3 Vectors, 1 Point }**, generating structures for bound particles, in addition to lines, planes, and volumes. Here, this is denoted as G3p1 .

With this full algebra, **Forces** from 2 or more particles can add to become a **Torque**, and two or more linear **Flows** can add to become a **Circulation** . This can not be expressed in the "tangent space" of Clifford Algebra Cl3, nor in standard Vector Algebra.

With similar clarity, the G3p1 algebra describes the motional "transformations" between vector electric $\vec{\mathrm{E}}$ and bi-vector magnetic $\widehat{\mathrm{B}}$, through 1st order ($\vec{v}$ / c) motion of an Origin Point, *without* 2nd order space-time algebra effects.

Thus, the algebra explicitly distinguishes between conduction currents and the (orthogonal) spin/circulation currents; and between dynamic effects such as "spin-transfer" torque, and entropic effects such as conduction resistance or magnetization damping.

Duality : It Takes Two to Tango

The triumph of Maxwell's Equations is the description of Electro-Magnetic waves, propagating outward from particles at fundamental speed **c** . These are generally viewed as either "light waves" or particle-like "Photons", expressing the "wave / particle" duality of modern physics.

Geometrically, there are two fundamental lengths in electro-magnetism :
(1) the "classical radius of the electron" $\mathbf{R_e} = e^2/m_e c^2$ = 2.82 pm (10^{-15} m), which scales the electric interaction field E between *two* or more electric charges; and
(2) the "Compton wavelength", here denoted $\mathbf{D_v} = \hbar c / m_e c^2$ = 386. pm, which scales the magnetic dipole field B from a *single* electron Spin.

In G3p1, the EM wave emanating from a single particle is seen to be nilpotent (a "Photino"), with energy densiy $\mathrm{E}^2 + \mathrm{B}^2 = 0$. However, the *superposition of two* counter-propagating Photinos is seen to give the inter-particle Force, Torque and Helicity transfers generally attributed to a single Photon.

Understanding the complete geometry of points, vectors, bi-vectors, and tri-vectors can substantially clarify the meaning of motional transforms in space and time, and of complex probability waves.

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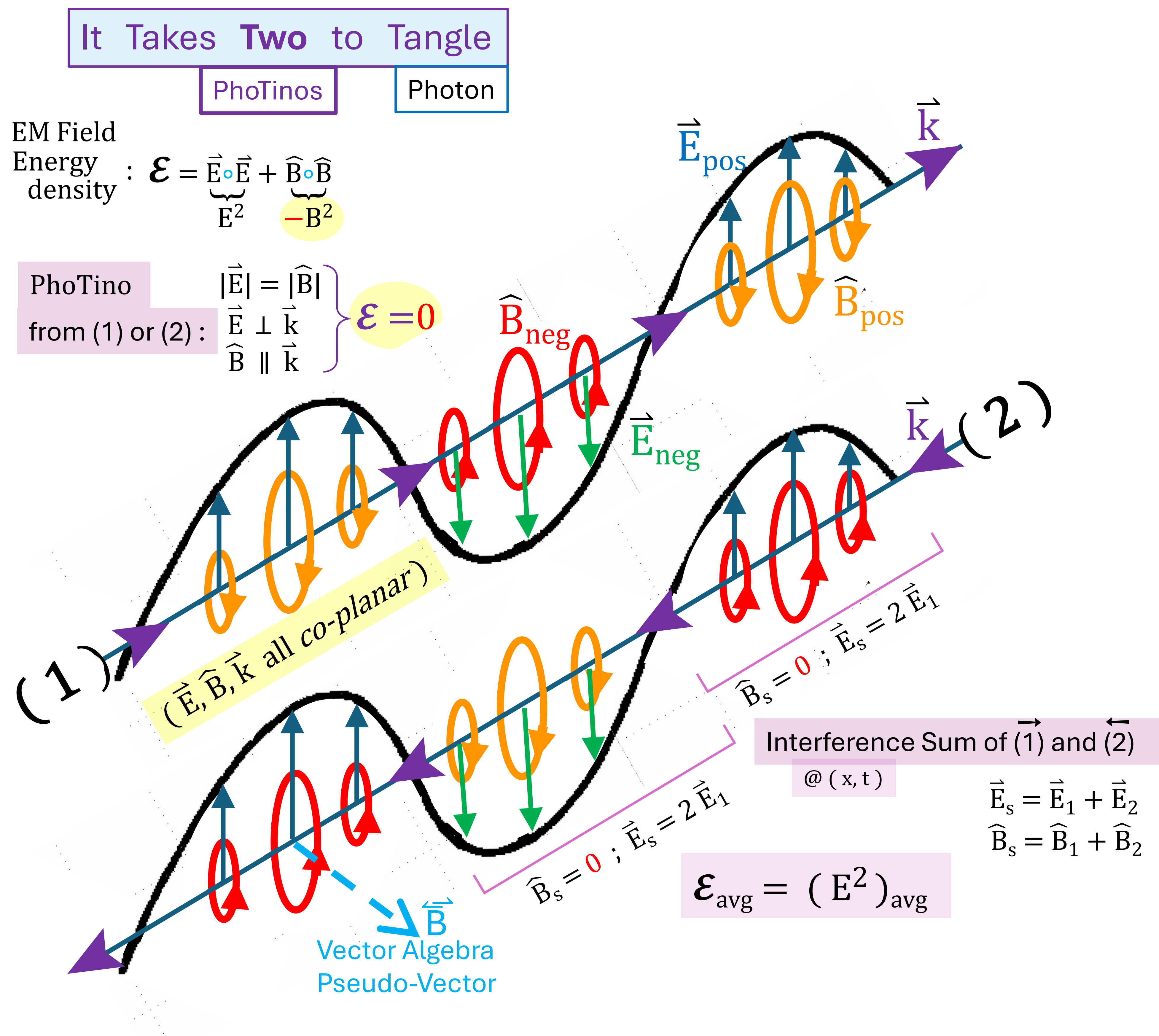
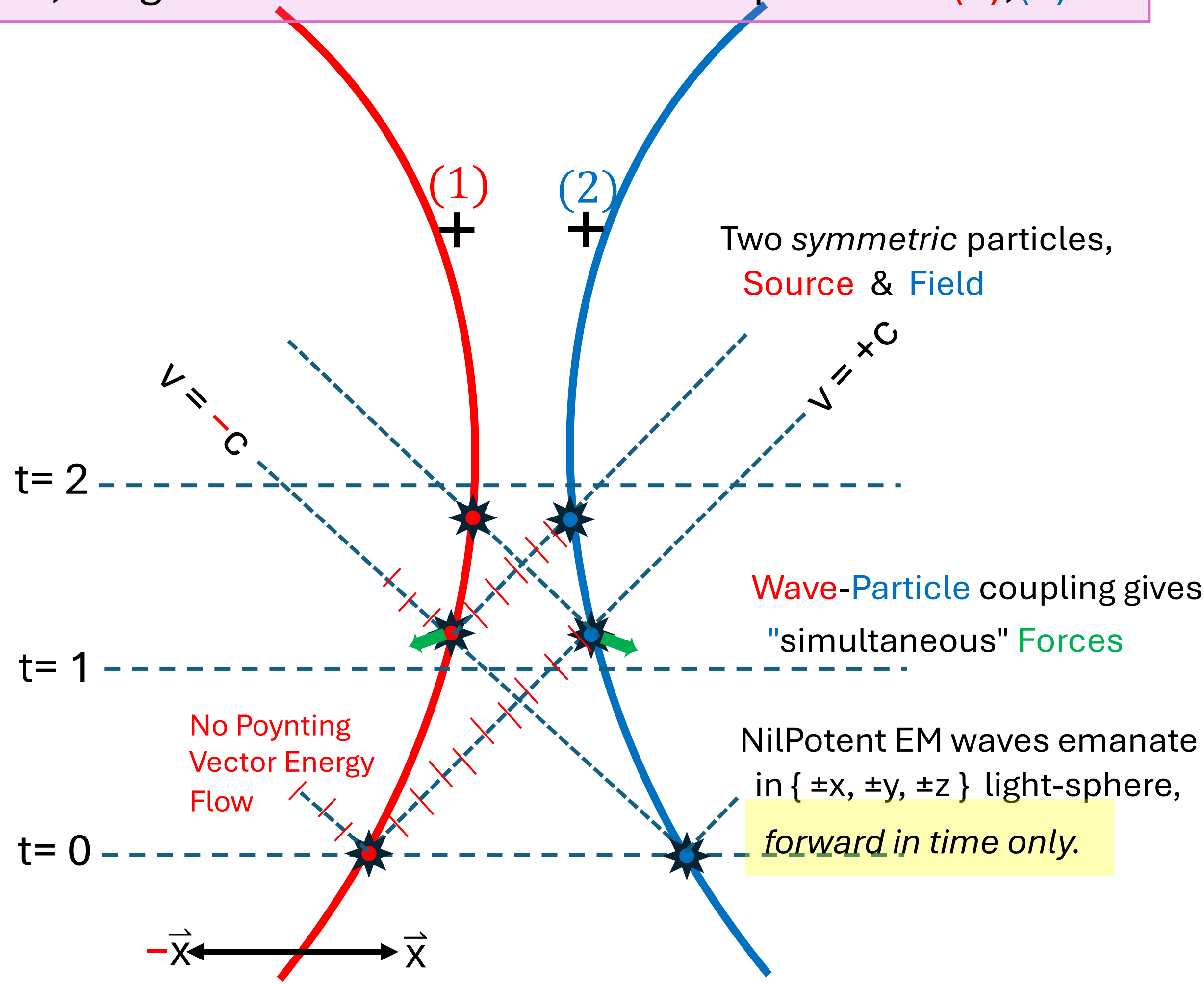
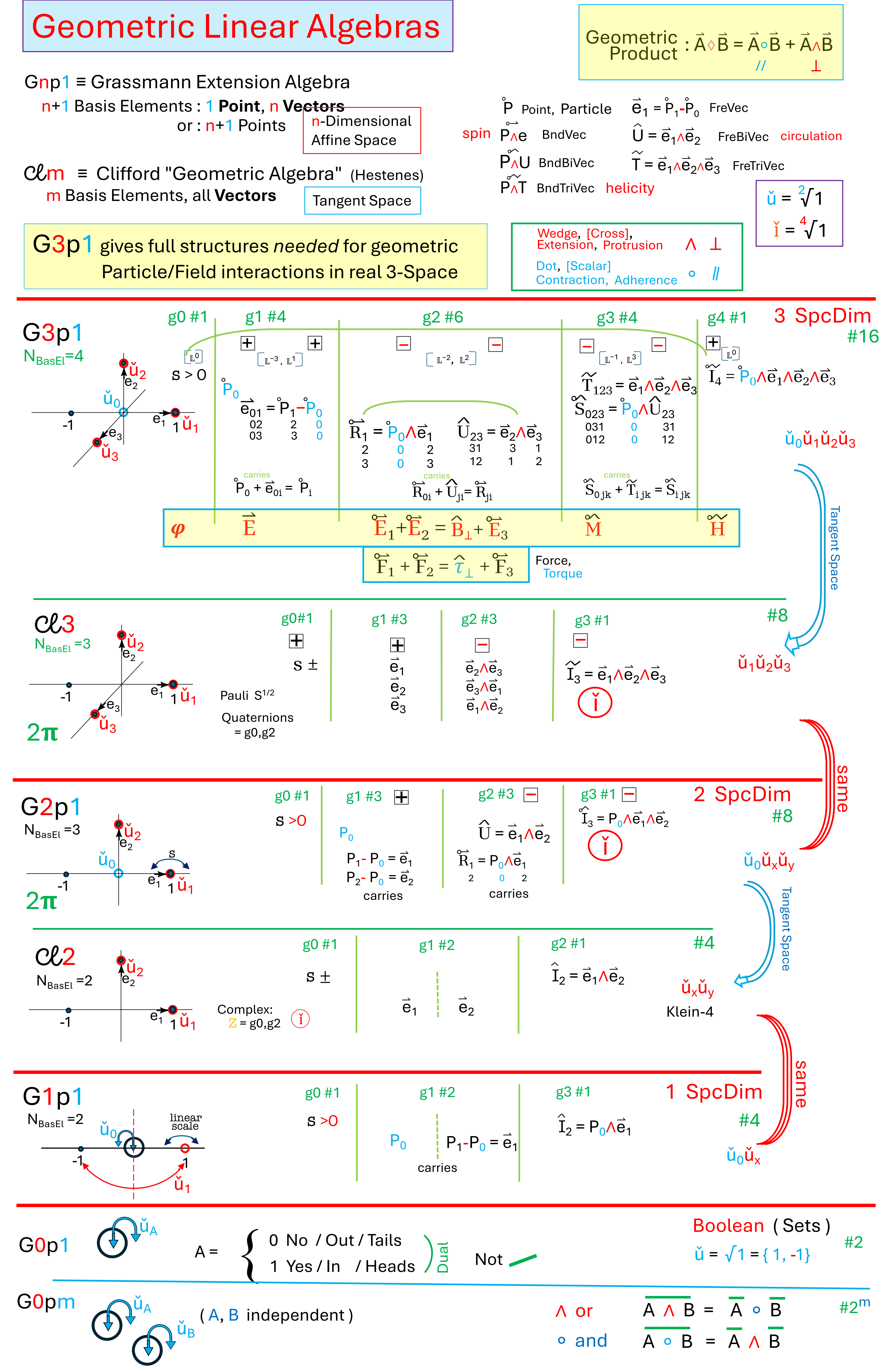

It Takes Two to Tangle
PhoTinos
Photon
EM Field Energy density : $\mathcal{E} = \underbrace{\vec{E} \circ \vec{E}}_{E^2} + \underbrace{\widehat{B} \circ \widehat{B}}_{-B^2}$
PhoTino from (1) or (2) :
$|\vec{E}| = |\widehat{B}|$
$\vec{E} \perp \vec{k}$
$\widehat{B} \parallel \vec{k}$
$\mathcal{E} = 0$
$\vec{E}_{pos}$
$\vec{k}$
$\widehat{B}_{pos}$
$\widehat{B}_{neg}$
$\vec{E}_{neg}$
$\vec{k}$
(2)
(1)
$(\vec{E}, \widehat{B}, \vec{k}$ all co-planar)
$\widehat{B}_s = 0 \; ; \; \vec{E}_s = 2\,\vec{E}_1$
$\widehat{B}_s = 0 \; ; \; \vec{E}_s = 2\,\vec{E}_1$
Interference Sum of $\overrightarrow{(1)}$ and $\overleftarrow{(2)}$
@ (x, t)
$\vec{E}_s = \vec{E}_1 + \vec{E}_2$
$\widehat{B}_s = \widehat{B}_1 + \widehat{B}_2$
$\mathcal{E}_{avg} = (E^2)_{avg}$
$\overleftrightarrow{B}$
Vector Algebra
Pseudo-Vector

Particles (1),(2) emanate "NilPotent" spherical waves w1,w2 ;
waves w1,w2 give simultaneous Forces on particles (1),(2) .





Geometric Linear Algebras
Geometric Product : A◇B = A∘B + A∧B
// ⊥
Gnp1 ≡ Grassmann Extension Algebra
n+1 Basis Elements : 1 Point, n Vectors
or : n+1 Points
n-Dimensional Affine Space
Clm ≡ Clifford "Geometric Algebra" (Hestenes)
m Basis Elements, all Vectors
Tangent Space
G3p1 gives full structures needed for geometric Particle/Field interactions in real 3-Space
P Point, Particle
spin P∧e BndVec
P∧U BndBiVec
P∧T BndTriVec helicity
e1 = P1-P0 FreVec
U = e1∧e2 FreBiVec circulation
T = e1∧e2∧e3 FreTriVec
ǔ = ²√1
ǐ = ⁴√1
Wedge, [Cross], Extension, Protrusion ∧ ⊥
Dot, [Scalar] Contraction, Adherence ∘ //
G3p1
NBasEl=4
g0 #1 g1 #4 g2 #6 g3 #4 g4 #1
3 SpcDim
#16
s > 0
P0
e01 = P1 - P0
02 2 0
03 3 0
P0 + e0i = Pi
R1 = P0∧e1
2 0 2
3 0 3
U23 = e2∧e3
31 3 1
12 1 2
R0i + Uji = Rji
T123 = e1∧e2∧e3
S023 = P0∧U23
031 0 31
012 0 12
S0jk + Tijk = Sijk
I4 = P0∧e1∧e2∧e3
ǔ0ǔ1ǔ2ǔ3
carries
φ E E1+E2 = B⊥+E3 M H
F1 + F2 = τ⊥ + F3
Force, Torque
Tangent Space
Cl3
NBasEl =3
2π
g0#1 g1 #3 g2 #3 g3 #1
#8
s ±
e1 e2 e3
e2∧e3 e3∧e1 e1∧e2
I3 = e1∧e2∧e3
ǐ
Pauli S^1/2
Quaternions = g0,g2
ǔ1ǔ2ǔ3
same
G2p1
NBasEl =3
2π
g0 #1 g1 #3 g2 #3 g3 #1
2 SpcDim
#8
s >0
P0
P1- P0 = e1
P2- P0 = e2
carries
U = e1∧e2
R1 = P0∧e1
2 0 2
carries
I3 = P0∧e1∧e2
ǐ
ǔ0ǔxǔy
Tangent Space
Cl2
NBasEl =2
g0 #1 g1 #2 g2 #1
#4
s ±
e1 e2
I2 = e1∧e2
Complex: Z = g0,g2 ǐ
ǔxǔy
Klein-4
same
G1p1
NBasEl =2
linear scale
g0 #1 g1 #2 g3 #1
1 SpcDim
#4
s >0
P0 P1-P0 = e1
carries
I2 = P0∧e1
ǔ0ǔx
G0p1
A = { 0 No / Out / Tails ; 1 Yes / In / Heads } Dual
Not
Boolean (Sets)
ǔ = √1 = { 1, -1}
#2
G0pm
(A, B independent)
∧ or
∘ and
A ∧ B = A ∘ B
A ∘ B = A ∧ B
#2^m

G3p1 linear algebra connects **Fields** to **Particle Scources** with **Charge** and **Spin**

$\mathcal{C}\ell 3$ Tangent	Fluid Maxwell (2-fluids)
g0 $\nabla\circ\vec{E} =$	$\nabla\circ\vec{E} = 4\pi(\rho_+ - \rho_-)$
g1 $-\nabla\circ\hat{B} =$	$\nabla\times\hat{B} = \frac{\partial}{\partial ct}\vec{E} + 4\pi\,\vec{J}/c$
g2 $\nabla\wedge\vec{E} =$	$\nabla\times\vec{E} + \frac{\partial}{\partial ct}\hat{B} = 0$
g3 $\nabla\wedge\hat{B} =$	$\nabla\circ\hat{B} = 0$

$e\boldsymbol{\varphi} \sim e^2\mathbb{L}^{-1} \sim \mathcal{E}$nergy

$\vec{E} \sim e\mathbb{L}^{-2}$

$\hat{B} \sim e\mathbb{L}^{-2} \sim e\mathbb{L}^{-1}\mathbb{T}^{-1}$

$\mathbb{L}$, $\mathbb{T}$ $\rangle$ c defines Time

$\mathbb{M}$ $\quad \mathcal{E}_0 \equiv m_e c^2$

Length scales of E & M

$R_e = \dfrac{e^2}{\mathcal{E}_0} \quad \sim 2.82\ 10^{-15}\ \text{m}$

$D_v = \dfrac{\hbar c}{\mathcal{E}_0} \quad \sim 386.\ 10^{-15}\ \text{m}$

G3p1

$\check{u} = \{1, -1\}$ $\qquad \check{u}_0\ \check{u}_x\ \check{u}_y\ \check{u}_z$

$q = \check{u}_Q\, e$

$\vec{r}_{sf} = \mathring{P}_s - \mathring{P}_f$

$R = |\vec{r}_{sf}|$

	e−	p+	e+	p−
$\check{u}_Q$	−	+	+	−
$\check{u}_H$	+	+	−	−

field @ (x_f, t) $\qquad$ source @ (x_s, τ)

g0 [$\mathcal{E}$] $\quad e\boldsymbol{\varphi} = e\int dV \sum\left\{ \dfrac{q_s\mathring{P}_s}{R} + \dfrac{e\mathring{P}_s}{R}\dfrac{p_{s\parallel}}{R} \right\}_{\tau = t-R/c}$

g1 [$e\mathbb{L}^{-3}$] $\quad \nabla\circ\vec{E} = 4\pi \sum\left\{ q_s\mathring{P}_s + e\mathring{P}_s\dfrac{p_{s\parallel}}{R} \right\}_{\tau = t-R/c}$

g1 $\quad \nabla\circ\hat{B} + \frac{\partial}{\partial ct}\vec{E} = 0$

g2 $\quad \nabla\wedge\vec{E} + \frac{\partial}{\partial ct}\hat{B} = 0$

Waves : Oscillate, Circulate

no $\vec{J}$

g3 $\quad \nabla\wedge\hat{B} = 4\pi \sum\left\{ \check{u}_s\mathring{P}_s \wedge \dfrac{\vec{M}_{s\parallel}}{R} \right\}_{\tau = t-R/c}$

g4 [$\mathcal{E}$] $\quad e\tilde{\boldsymbol{\mathcal{H}}} = e\int dV \sum\left\{ \check{u}_s\mathring{P}_s \wedge \dfrac{\vec{M}_{s\parallel}}{R} \right\}_{\tau = t-R/c}$

1D-Dipole Charge

$\vec{p}_s = (\mathring{P}_{s+} - \mathring{P}_{s-}) \quad [\mathbb{L}^1]$

$p_{s\parallel} = \vec{p}_s \circ \vec{r}^{\,u}_{sf}$

2D-Dipole Magnetic Spin

$\hat{M}_s = \frac{1}{2}\, D_v\, c\, \hat{S}^{u} \quad [\mathbb{L}^2\mathbb{T}^{-1}]$

$\vec{M}_{s\parallel} = \hat{M}_s \circ \vec{r}^{\,u}_{sf}$

with $\hat{S}^{u}_s = \check{u}_H(\vec{\beta}^{u}_s \circ \tilde{T}^{u})$

i.e. $\hat{S}_s \perp \vec{\beta}_s$

Positional Energy $\qquad$ Spin 2D-Dipole Orientation $\qquad$ Charge 1D-Dipole Orientation

$\mathcal{E}/\mathcal{E}_0 = q_f\left[\boldsymbol{\varphi}(x_f) - \hat{M}_f \circ \hat{B} + \vec{d}_f \circ \vec{E}\right]$

Particle Force

$\vec{F}_f/\mathcal{E}_0 = q_f\left(\vec{E} - \hat{M}_f \circ \nabla\hat{B} + \vec{d}_f \circ \nabla\vec{E}\right)$

Particle Torque

$\hat{\boldsymbol{\tau}}_f/\mathcal{E}_0 = q_f\left(- \hat{M}_f \wedge \hat{B} + \vec{d}_f \wedge \vec{E}\right)$

EM Waves :

$\omega / \vec{k} = c$

$\tilde{f} = \sin(\vec{k}\circ\vec{r} - \omega t)[\,\vec{E} + \hat{B}\,]$

$\vec{k} \perp \vec{E}, \quad \hat{B} \parallel \vec{k}$

Invariants :

$\boldsymbol{\mathcal{E}} = \vec{E}^2 + \hat{B}^2 \quad$ Energy

$\hat{\boldsymbol{\mathcal{H}}} = \vec{E} \wedge \hat{B} \quad$ Helicity

Nilpotent "Photino"

$\hat{B} = \check{u}_H\, \vec{k} \wedge \vec{E} \qquad \boldsymbol{\mathcal{E}} = 0$

co-planar : $\hat{B} \parallel \vec{E}$